Technical Comments

Comment on "Applications of Bolotin's Method to Vibrations of Plates"

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IN a recent interesting Technical Note,[1] King and Lin applied Bolotin's asymptotic method to the determination of natural frequencies of rectangular plates. The writer agrees with their observation that little attention to this method has been given in Western literature. The main purpose of this Comment is to mention earlier work, where the method was used to determine the natural frequencies of plate systems.[2]

In Ref. 2, the writer applied this method to single- and multi-plate systems and gave approximate frequency equations for 2 two-plate systems, where the two plates were in perpendicular planes and were joined along a common edge. The first ten natural frequencies for each of these systems were obtained and agreed to within 0.5% with a solution based on a series method; the latter has been described in more detail elsewhere[3] and was based on the work of Dill and Pister.[4] A few frequencies were obtained for a closed box using Bolotin's method; again good agreement was obtained with frequencies from the series method. However, Bolotin's method has limitations, as it predicts mode shapes which have nodal lines parallel to the sides of the plates. For the two-plate systems considered in Ref. 2 some of the modes could not easily be approximated in this way, but nevertheless the Bolotin method gave accurate natural frequencies. For box-type structures many modes have nodal patterns which cannot be represented by lines parallel to the plate edges and here Bolotin's method failed to predict the natural frequencies. This is one disadvantage of the method. Its other disadvantage is that frequencies of increasing accuracy cannot be obtained by increased computational effort; i.e., there is no procedure similar to the use of additional terms or a finer mesh in the series, Rayleigh-Ritz, finite difference, and finite element methods. If the writer had to determine natural frequencies of plate systems today, he would use the finite element method because of its versatility and the availability of a general program; the natural frequencies of the box-type structures of Refs. 2 and 3 have been obtained satisfactorily by this method.[5] However, for simple-plate systems, a useful check on the accuracy of frequencies by the finite element method could be provided by both the series method and Bolotin's method, particularly for higher modes.

The second purpose of this comment is to remark upon some of the natural frequencies in Table 3 of Ref. 1. King and Lin state that they were unable to find in the literature a value of the natural frequency for the sixth mode of a square plate with two adjacent edges clamped and the other two free to compare with that obtained by Bolotin's method. From Leissa's recent paper,[6] the value of the nondimensional frequency factor for this mode is 65.833. He used 36 terms in a Rayleigh-Ritz solution. Thus the frequency factors of Ref. 6 are more accurate than Young's values, which are quoted in Table 3 and for which a smaller number of terms was used. As we mentioned, Bolotin's method predicts mode shapes with nodal lines parallel to the edges of the plate. Thus it gives identical frequencies for the second and third modes of this plate and also for the fifth and sixth modes, as pointed out by King and Lin. In this way it is similar to the use of the Rayleigh-Ritz method with a single term representing the assumed vibration function. However, it is relatively simple with the Rayleigh-Ritz method to obtain reasonably accurate discrete natural frequencies for these pairs of modes by using two appropriate terms in the assumed functions, as shown earlier by the writer.[7]

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References

[1] King, W. W. and Lin, C.-C., "Applications of Bolotin's Method to Vibrations of Plates," *AIAA Journal*, Vol. 12, March 1974, pp. 399–401.

[2] Dickinson, S. M. and Warburton, G. B., "Natural Frequencies of Plate Systems Using the Edge Effect Method," *Journal of Mechanical Engineering Science*, Vol. 9, Dec. 1967, pp. 318–324.

[3] Dickinson, S. M. and Warburton, G. B., "Vibration of Box-Type Structures," *Journal of Mechanical Engineering Science*, Vol. 9, Dec. 1967, pp. 325–335.

[4] Dill, E. H. and Pister, K. S. "Vibration of Rectangular Plates and Plate Systems," *ASME Proceedings of the 3rd U.S. National Congress of Applied Mechanics*, 1958, pp. 123–132.

[5] Henshell, R. D., "Transmission of Vibration in Damped Elastic Structures," Ph.D. thesis, May 1967, Department of Mechanical Engineering, University of Nottingham, Nottingham, U.K.

[6] Leissa, A. W., "The Free Vibration of Rectangular Plates," *Journal of Sound and Vibration*, Vol. 31, Dec. 1973, pp. 257–293.

[7] Warburton, G. B., "The Vibration of Rectangular Plates," *Proceedings of the Institution of Mechanical Engineers*, Vol. 168, 1954, pp. 371–381.

Comment on "Variations of Eigenvalues and Eigenfunctions in Continuum Mechanics"

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FARSHAD[1] has presented the formal equations for first-order perturbation theory as applied to a fairly general class of linear differential equations. The general theory, as first systematically formulated and applied by Rayleigh,[2] has been mainly applied to the self-adjoint differential equations of conservative physical systems, and in this application is very well known in the literature of physics and applied mathematics for both continuous and discrete systems.[3–8] Farshad's illustrative example of the longitudinal vibration of a bar of uniform structural properties but nonuniform mass distribution, is of the self-adjoint type and is thus already extremely well represented in the literature.

Unfortunately, Farshad[1] in his illustrative example has made mathematical errors in computing the first-order changes in both the eigenvalues and the eigenfunctions. Although the first-order change in the eigenvalues is unaffected by a first-order change in the eigenfunctions, the eigenfunctions do undergo a first-order change with perturbation of the parameters of a vibrating system unless the changes in the system are such that the exact eigenfunctions (normal modes) of the perturbed system are identical to those of the unperturbed system. Clearly such is not the case

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in Farshad's example and it is apparent that his calculation of the eigenfunction series coefficients a_{mj} are in error since they cannot be all zero for $m \neq j$. This would lead to unchanged normal modes to all orders in perturbation theory. Straightforward integration of the expression for the coefficients verifies that they are not zero. In arriving at the first-order perturbations of the eigenfunctions here, we will not employ a series expansion in the unperturbed eigenfunctions, but rather a more direct method which appears to be less well known than the series expansion method although it is often more convenient.

Farshad's illustrative problem is governed by the differential equation

$$y'' + \lambda^2(1+\varepsilon x)y = 0 \tag{1}$$

For $\varepsilon = 0$, this equation is well known[2,3] and arises, for example, both in the longitudinal and torsional vibration of a uniform slender bar held at one end and free at the other.[2] The eigenvalues are $\lambda_{n0} = (2n+1)\pi/2$, where n takes on integral values from zero to infinity. The corresponding eigenfunctions (non-normalized) are $y_{n0} = \sin \lambda_{n0} x$.

In perturbation theory it is assumed that the perturbed eigenvalues and eigenfunctions are given by power series of the form (suitable for the present case)

$$\begin{aligned} \lambda_n{}^2 &= \lambda_{n0}{}^2 + \varepsilon\lambda_{n1}{}^2 + \varepsilon^2\lambda_{n2}{}^2 + \cdots \\ y_n &= y_{n0} + \varepsilon y_{n1} + \varepsilon^2 y_{n2} + \cdots \end{aligned} \tag{2}$$

The first-order variation in the eigenvalues for small values of ε is, from the standard formulas of small perturbation theory,[3]

$$\lambda_{n1}{}^2 = -\frac{\lambda_n{}^2 \int_0^1 x \sin^2 \lambda_n x \, dx}{\int_0^1 \sin^2 \lambda_n x \, dx} \tag{3}$$

This is readily evaluated to give

$$\lambda_{n1}{}^2 = -1/2\{\lambda_{n0}{}^2 + 1\} = -1/2\{(2n+1)^2\pi^2/2^2 + 1\} \tag{4}$$

which is not in agreement with Farshad's result.

The differential equation for the first-order perturbation in the eigenfunctions may be obtained[6] by substitution of Eq. (2) into the differential Eq. (1), retaining the first-order terms in ε. This leads to

$$y_{n1}'' + \lambda_{n0}{}^2 y_{n1} = -\lambda_{n0}{}^2 x y_{n0} - \lambda_{n1}{}^2 y_{n0} \cdots \tag{5}$$

with boundary conditions $y_{n1}(0) = y_{n1}'(1) = 0$.

This is formally a non-homogeneous equation which has a "resonance" character (i.e., a forcing term with a component, y_{n0}, corresponds to the eigenvalue $\lambda_{n0}{}^2$ and gives rise to a theoretically infinite response). It is well known[9] that such an equation has a unique solution, if, and only if

$$\int_0^1 f(x) y_{n0}(x)\, dx = 0 \cdots \tag{6}$$

where $f(x)$ is the "forcing function" on the right-hand side of Eq. (5). This condition, in effect, requires that the y_{n0} modal component be zero.

In the present case $f(x) = \lambda_{n1}{}^2 y_{n0} - \lambda_{n0}{}^2 x y_{n0}$ and Eq. (6) is automatically satisfied by virtue of Eq. (3) for $\lambda_{n1}{}^2$, the first-order perturbation in $\lambda_n{}^2$. Thus we must solve the nonhomogeneous differential equation

$$y_{n1}'' + \lambda_{n0}{}^2 y_{n1} = -\lambda_{n0}{}^2 x \sin \lambda_{n0} x - \lambda_{n0}{}^2 \sin \lambda_{n0} x \cdots \tag{7}$$

The solution may be found by elementary methods to be

$$y_{n1} = \lambda_{n0}(x^2/4)\cos \lambda_{n0} x - (x/4)\sin \lambda_{n0} x + (\lambda_{n1}{}^2/2\lambda_{n0}) x \cos \lambda_{n0} x \cdots \tag{8}$$

It is obvious that this solution satisfies the boundary condition $y_{n1}(0) = 0$; it may easily be verified that the boundary condition $y'_{n1}(1) = 0$ is satisfied by substituting for λ_{n1} from Eq. (4).

The next step in the perturbation process would be to compute the eigenvalues correct to the second order from the first order perturbed eigenfunctions previously given. Or, alternatively, the assumed eigenfunctions, $y_{n0} + \varepsilon y_{n1}$, may be substituted into the Rayleigh quotient, which usually will lead to a more accurate approximation. However, if only the lowest eigenvalue is of interest, it should be noted that the first step of the iteration-variation method[5] gives excellent results with relatively little effort and is usually simpler to use and more accurate than formal perturbation theory.

References

[1] Farshad, M., "Variations of Eigenvalues and Eigenfunctions in Continuum Mechanics," *AIAA Journal*, Vol. 12, April 1974, pp. 560–561.

[2] Rayleigh, Lord, *The Theory of Sound*, Vol. I, Dover, New York, 1945, pp. 113–118, 136–137.

[3] Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Vol. I, Interscience, New York, 1953, pp. 41–45, 343–350.

[4] Titchmarsh, E. C., *Eigenfunction Expansions*, Part II, Oxford, New York, 1958, pp. 212–257.

[5] Morse, P. M. and Feshbach, H., *Methods of Theoretical Physics*, Vol. II, McGraw-Hill, New York, 1933, pp. 1001–1064, 1106–1112.

[6] Collatz, L., *Eigenwert Probleme und Ihre Numerische Behandlung*, Chelsea Publ. Co., New York, 1948, pp. 305–313.

[7] Lancaster, P., *Lambda Matrices and Vibrating Systems*, Pergamon, New York, 1966, pp. 28–37, 73–76, 168–183.

[8] Morse, P. M., *Vibration and Sound*, McGraw-Hill, New York, 1948, pp. 122–133, 164–165.

[9] Ince, E. L., *Ordinary Differential Equations*, Dover, New York, 1944, pp. 213–214, 266–269.